Consistent Measures of Risk*

Jón Daníelsson
London School of Economics
j.danielsson@lse.ac.uk

Bjørn N. Jorgensen
Columbia Business School
bnj2101@columbia.edu

Mandira Sarma
Indian Statistical Institute
sarma_mandira@vsnl.net

Casper G. de Vries
Erasmus University
cdevries@few.eur.nl

Jean-Pierre Zigrand
London School of Economics
j.p.zigrand@lse.ac.uk

May 2008

Abstract

We characterize the partial orderings induced by the most common risk measures and compare them to the partial orderings induced by first and second order stochastic dominance, respectively. We show which risk measures are consistent in the sense that they induce the same partial orderings as stochastic dominance. We also demonstrate which risk measures exhibit the property that stochastic dominance among risky choices imply consistency, and whether the reverse is true. Finally, we find that tail conditional expectation does not meet these consistency criteria.

KEY WORDS: stochastic dominance, risk measures, preference ordering, utility theory

JEL Classification: D81, G00, G11

*Corresponding author Jean-Pierre Zigrand, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom. Our papers can be downloaded from `www.RiskResearch.org`. J. Danielsson and J.-P. Zigrand would like to acknowledge financial support under the EPSRC Grant GR/S83975/01 at the Financial Markets Group, London School of Economics. We also benefited from the comments of an anonymous referee.

1 Introduction

A consequence of the many risk measures in common usage is that they may provide conflicting guidance for selection among investment choices. Consensus of how to discriminate among risk measures is lacking. In contrast, the academic literature has long agreed that maximisation of expected utility functions, and the stochastic dominance criteria related to utility, lead to appropriate decision rules. Our objective is to formally apply the stochastic dominance criteria to risk measures in common usage. Specifically, we investigate whether each risk measure leads to the same ordering of investment choices as first and second order stochastic dominance, respectively.

Spurned on by financial supervisors, the financial industry employs statistical risk measures to evaluate the riskiness of investments motivated by the desire to measure risk without necessarily having to model the underlying distribution of financial returns. Based on historical observations of returns, these risk measures offer summary statistics describing returns, such as a quantile (i.e., Value-at-Risk, VaR) and various forms of conditional expectations of extreme events, of which expected shortfall is one of many alternatives. Obviously, when used to rank alternative investment choices, different risk measures may provide conflicting advice.

Similarly, different utility functions can lead to different investment decisions. However, when investment choices can be ranked using stochastic dominance criteria, the preferred investment choice would remain the same for a large class of utility functions. Specifically, first (second) order stochastic dominance of one random payoff over another is equivalent to saying that any investor with non-decreasing (concave) utility prefers the former.[1] We relate common risk measures to expected utility maximization through the stochastic dominance criteria.

Consider two random variables X and Y, X first and second order stochastically dominates Y, denoted by $X\text{FSD}Y$ and $X\text{SSD}Y$, respectively.[2] Further, we say that X is less risky than Y, denoted by $X\text{LR}Y$ if $X\text{SSD}Y$ and the

[1]This first order stochastic dominance result is originally due to Quirk and Saposnik (1962) and Hanoch and Levy (1969) for compact random payoffs. In subsequent work, Brumelle and Vickson (1975) and Tesfatsion (1976) extended these results to unbounded random variables provided an integrability condition is satisfied. The second order stochastic dominance result is originally from Hanoch and Levy (1969) and Rothschild and Stiglitz (1970). The more general case of random payoffs with unbounded support requires another integrability condition, see Tesfatsion (1976). Throughout this paper we assume that both integrability conditions obtain.

[2]Formally, let F denote the distribution function. Then $X\text{FSD}Y$, if $F_X(z) \leq F_Y(z) \quad \forall z$, where $z \in S \subseteq \mathbb{R}$. Further, $X\text{SSD}Y$, if

random variables have the same mean. Below, we denote by XSDY that X stochastically dominates Y, where SD is either FSD, SSD or LR.

The partial ordering induced by LR is particularly useful in our setting for by controlling for the mean, it solely compares the "risk" of X and Y, whereas the concept of SSD may involve an amalgamation of both risk and expected return considerations. LR has been introduced in the seminal paper by Rothschild and Stiglitz (1970). When comparing X and Y with different means, for risk comparison purposes one can always add the right amount of the riskless asset to the asset with the lower mean and then compare them using LR. So while SSD is more general in principle, for risk comparisons LR may be more useful.

We next introduce three definitions of consistency. A risk measure ρ is

SD>consistent if XSDY then $\rho_X \leq \rho_Y$

SD<consistent if $\rho_X \leq \rho_Y$ then XSDY

SD-consistent if ρ is both SD>consistent and SD<consistent.

Risk measures can be parameterized by a variable $\pi \in \Pi$, typically a reference point such as a cutoff quantile or probability level, in which case $\rho_X \leq \rho_Y$ means $\rho_X(\pi) \leq \rho_Y(\pi)$, all $\pi \in \Pi$. Further, we say that ρ is *partially* SD>*consistent* if XSDY implies $\rho_X(\pi) \leq \rho_Y(\pi)$ over a subset of Π.

Of our three notions of consistency, SD<consistency is of great importance to financial institutions. Suppose a risk manager relies on some risk measure ρ which is *not* SD<consistent. Even though he might be confident that X is less risky than Y for a large set of reference points, say all of Π, and that $E[X] = E[Y]$, he would nevertheless not be able to conclude that the owners would necessarily agree with his choice of X over Y. For SD<consistent ρ he would have that assurance.

A major motivation for studying SD>consistency of risk measures is the large and growing literature on risk measures as decision tools in asset allocation. Markowitz (1959), among many others, use the standard deviation, σ, as risk measure such that $\mu - \sigma$ defines the efficient set, μ being the mean. The idea is to replace the standard deviation by another risk measure, ρ, and investigate the $\mu - \rho$ efficient set. For example, Kaplanski and Kroll (2002) study the $\mu -$ VAR and $\mu -$ ES efficient sets, where ES is expected

$$\int_{-\infty}^{q} F_X(z)dz \leq \int_{-\infty}^{q} F_Y(z)dz, \quad \forall q \in S \tag{1}$$

shortfall. A portfolio is a member of this efficient set iff it is not dominated by any other porfolio (i.e. any other portfolio having lower ρ risk also has lower expected payoff). Establishing that ρ is SD>consistent is equivalent to establishing that the $\mu - \rho$ efficient set is a subset of the SD efficient set. In other words if a portfolio is efficient by the $\mu - \rho$ criterion, then it also is efficient by the SD criterion. This would indicate that portfolio allocation by a $\mu - \rho$ criterion should only be attempted with risk measures that are SD>consistent.

We consider the following risk measures: Value–at–Risk (VaR) , tail conditional expectation (TCE) (or "TailVaR"), expected shortfall (ES), lower partial moments (LPM) of the zeroth (ZLPM), first (FLPM) and second order (SLPM) (see Bawa, 1975), as well as the Omega function (Keating and Shadwick, 2002).[3] First, we show that ZLPM and VaR are FSD-consistent. Second, we demonstrate that FLPM, SLPM, ES and Omega are FSD>consistent, but not FSD<consistent. Third, we show that TCE is neither FSD>consistent nor FSD<consistent. These findings extend Fishburn (1977) who shows that LPM retains a FSD>consistent ordering and Kaplanski and Kroll (2002) who provide sufficient conditions such that VaR is FSD>consistent.

Fourth, we show that for arbitrary distributions, ES, Omega and FLPM are LR-consistent. In contrast, standard deviation, SLPM, ZLPM, VaR and IQR are LR>consistent, but not LR<consistent, with ZLPM and VaR partially below the first crossing point only and with IQR under some conditions. For two-parameter distributions, the standard deviation, IQR and β are LR-consistent. Again, TCE is neither LR>consistent nor LR<consistent, unless the distribution functions satisfy certain continuity conditions.[4] The last finding would suggest that TCE is best avoided since it fails all consistency criteria as well as coherence.

2 Risk measures

This paragraph introduces the notation. Suppose that X and Y are two risky asset returns with distribution functions F_X and F_Y respectively. A point of discontinuity corresponds to a point with strictly positive mass. The upper p-quantile $q_X(p)$ is defined as $\sup\{x : F_X(x) \leq p\}$. The lower p-quantile is

[3]ES is a generalization of $\text{TCE}_X := -E[X|X \leq -\text{VaR}_X]$ due to Acerbi and Tasche (2002) which restores coherence of the risk measure in the sense of Artzner et al. (1999).

[4]Yoshiba and Yamai (2002) show LR>consistency when distribution functions admit densities.

the generalized inverse function $\tilde{F}_X^{-1}(p) := \inf\{x : F_X(x) \geq p\}$. If X and Y are integrable, X and Y have expected values $\mu_X = E[X] = \int_{\mathbb{R}} x dF_X(x)$ and $\mu_Y = E[Y] = \int_{\mathbb{R}} y dF_Y(y)$. If second moments exist, we denote by σ_X and σ_Y the standard deviations of X and Y respectively.

Dhaene et al. (2003) classify risk measures based on whether they consider the entire set of outcomes, referred to as *overall risk measures*, or only the tails, the so-called *downside risk measures*. A risk measure is any mapping from the relevant space of financial risk to the real line.

Among the overall risk measures, we distinguish:

1. Variance σ_X^2 (provided $X \in L_2$), is given by

$$\sigma_X^2 := \int_{-\infty}^{\infty} (x - \mu_X)^2 dF_X(x)$$

2. Market risk β_X is given by

$$\beta_X := \rho_{X,R} \ \frac{\sigma_X}{\sigma_R}$$

   where σ_R is the standard deviations of the market portfolio R and $\rho_{X,R}$ is the correlation coefficient between X and R.

3. The interquartile range (IQR) measure reads

$$\text{IQR} = q_X(3/4) - q_X(1/4)$$

The IQR measure is sometimes used as a measure of overall risk when the second moment is not bounded. For example, for symmetric α-stable distributions with $1 < \alpha < 2$ the standard deviation does not exist, but the scale can be captured by IQR (Fama and Roll, 1968).

Among the downside risk measures, we consider:

1. Second Lower Partial Moment (SLPM) is defined as[5]

$$\text{SLPM}(q) := \int_{-\infty}^{q} (q - x)^2 dF_X(x) \quad = \quad 2 \int_{-\infty}^{q} (q - x) F_X(x) dx$$

   assuming that $\int_{-\infty}^{0} x^2 dF_X(x) < \infty$.

[5]The second equality follows from integrating by parts, using the fact that $\int_{-\infty}^{0} x^2 dF_X(x) < \infty$ implies $\lim_{x \to -\infty} x^2 F_X(x) = 0$ and the fact that $\int_{-\infty}^{0} x^2 dF_X(x) < \infty$ implies that $\int_{-\infty}^{0} x dF_X > -\infty$, which in turn implies that $\lim_{x \to -\infty} x F_X(x) = 0$.

2. First Lower Partial Moment (FLPM):

$$\text{FLPM}(q) := \int_{-\infty}^{q} (q-x)dF_X(x) \quad = \quad \int_{-\infty}^{q} F_X(x)dx$$

where the equality follows again from integration by parts, provided $\int_{-\infty}^{0} xdF_X > -\infty$.

3. Zeroth Lower Partial Moment (ZLPM):

$$\text{ZLPM}(q) := \int_{-\infty}^{q} dF_X = F_X(q)$$

4. Value-at-Risk (VaR): If the ZLPM(q) is fixed at p, then the negative of the upper quantile gives the Value-at-Risk as

$$\text{VaR}_X(p) := -q_X(p)$$

VaR is defined as the maximum potential loss to an investment with a pre-specified confidence level $(1-p)$.

5. Tail Conditional Expectation (TCE) at the confidence level $(1-p) < 1$ is defined as

$$\begin{aligned}\text{TCE}_X(p) &:= -E[X|X \leq -\text{VaR}_X(p)] \\ &= -\int_{-\infty}^{q_X(p)} z \ \frac{dF_X(z)}{F_X(q_X(p))} = -q_X(p) + \frac{1}{F_X(q_X(p))}\int_{-\infty}^{q_X(p)} F_X(z)dz\end{aligned}$$

The last equality (provided $p > 0$ and $\int_{-\infty}^{0} xdF_X > -\infty$) follows from integration by parts. Some authors and most practitioners call this risk measure "expected shortfall" (ES). We follow Artzner et al. (1999) by calling it TCE and follow Acerbi et al. (2001) by reserving the term ES for the following variant.

6. Expected Shortfall (ES) at the confidence level $(1-p) < 1$ is defined as

$$\begin{aligned}\text{ES}_X(p) &:= -\frac{1}{p}\Big(\int_{-\infty}^{q_X(p)} zdF_X(z) - q_X(p)[F_X(q_X(p)) - p]\Big) \\ &= \text{TCE}_X(p) + \frac{F_X(q_X(p)) - p}{p}[\text{TCE}_X(p) - \text{VaR}_X(p)] \\ &= -\frac{1}{p}\int_{0}^{p} \tilde{F}_X^{-1}(z)dz\end{aligned}$$

The last equality is due to Acerbi and Tasche (2002)[6] who also show that ES_X is in fact identical to the risk measure known as "Conditional Value-at-Risk." Notice that when F_X is continuous, then $F_X(q_X(p)) = p$ and ES and TCE coincide.[7] If on the other hand F_X is not continuous and $p \notin F_X(\mathbb{R})$, then by definition of $q_X(p)$ the event $\{X \leq q_X(p)\}$ has probability $F_X(q_X(p)) > p$, in which case TCE might violate subadditivity. To ensure subadditivity (and consistency, as we show later), the relevant amount has been substracted, and some authors (e.g. Acerbi and Tasche (2002)) reserve the term ES for this subadditive transformation of TCE, as we do here. Also, notice that $\mathrm{ES}_X(p) \geq \mathrm{TCE}_X(p)$.

7. Omega: (Ω) is a risk measure[8] defined as (with the obvious adjustments if the denominator is zero)

$$\Omega_X(q) := \frac{\int_{-\infty}^{q} F_X(z)dz}{\int_{q}^{\infty}(1 - F_X(z))dz}$$

3 Risk measures and first order stochastic dominance

It is well–known that XFSDY implies that $\mu_X \geq \mu_Y$, and that the converse is not true in general. The case for a risk measure to satisfy FSD>consistency is therefore weak since FSD is a much stronger concept that goes beyond risk and pertains to expected returns also. It is also well-known that XFSDY does not lead to an unambiguous ordering between assets with respect to any of the overall risk measures. One would, however expect to be able to say something about the FSD>consistency of downside measures because less downside risk tends to mean more upside risk, which is of relevance to a non–satiated investor.

Some results on downside measures already exist in the literature. From Fishburn (1977) and Kaplanski and Kroll (2002), we know that the ordering of investment choices with respect to SLPM, FLPM, ZLPM and

[6]These authors also show that ES is not quantile dependent in the sense that in the definition of ES, the upper quantile $q_X(p)$ can be replaced by any $\alpha \in [\tilde{F}_X^{-1}(p), q_X(p)]$ without affecting the function ES.

[7]More generally, ES and TCE coincide iff for each $p \in [0,1]$, either $F_X(q_X(p)) = p$ or $\mathbb{P}(X < q_X(p)) = 0$.

[8]Due to Keating and Shadwick (2002), Omega has originally been designed as the performance measure $\frac{\int_q^\infty (1-F_X(z))dz}{\int_{-\infty}^q F_X(z)dz}$, balancing upside potential and risks.

VaR is FSD>consistent. FSD<consistency of ZLPM and VaR can be found in Föllmer and Schied (2002). We complement these results by deriving FSD−consistency for ES and Omega, by showing that TCE is neither FSD>consistent nor FSD<consistent. Finally we completely characterize consistency of global risk measures for two-parameter distributions. The statement $\text{TCE}_X \; ? \; \text{TCE}_Y$ means that for arbitrary distribution functions we are neither assured that $\text{TCE}_X \leq \text{TCE}_Y$ nor the other way around, and that there are cases of F_X and F_Y whereby $\text{TCE}_X(p) < \text{TCE}_Y(p)$ for some p and $\text{TCE}_X(p) > \text{TCE}_Y(p)$ for other p. All proofs are relegated to the appendix.

Proposition 1 *Regardless of the distributions of X and Y, the following statements are equivalent:*

1. $X\text{FSD}Y$
2. $\text{ZLPM}_X \leq \text{ZLPM}_Y$
3. $\text{VaR}_X \leq \text{VaR}_Y$

If $X\text{FSD}Y$ then the following inequalities hold, while the converse is false:

$$\text{SLPM}_X \leq \text{SLPM}_Y \quad (2)$$
$$\text{FLPM}_X \leq \text{FLPM}_Y \quad (3)$$
$$\text{ES}_X \leq \text{ES}_Y \quad (4)$$
$$\Omega_X \leq \Omega_Y \quad (5)$$
$$\text{TCE}_X \; ? \; \text{TCE}_Y \quad (6)$$

TCE may therefore be neither coherent (Artzner et al. (1999)) nor FSD>consistent, while ES is both coherent and FSD>consistent.

Much of actual asset allocation and performance evaluation in wealth management is still performed under the simplifying assumptions of two-parameter return densities over the entire real line. The following proposition is useful in further characterizing FSD in such a world:

Proposition 2 *Assume now that X and Y are such that $F_X(x) = F_Y(y)$ whenever $\frac{x-\mu_X}{\sigma_X} = \frac{y-\mu_Y}{\sigma_Y}$. Also, assume that F_X and F_Y admit strictly positive densities over $\mathbb{R}$. The following statements are equivalent:*

a. $X\text{FSD}Y$

b. $\sigma_X = \sigma_Y$ *and* $\mu_Y \leq \mu_X$

c. $\text{IQR}_X = \text{IQR}_Y$ *and* $\mu_Y \leq \mu_X$

4 Risk measures and second order stochastic dominance

In this section we compare orderings induced by the risk measures to the SSD ordering. The following equivalence results can be shown:

Proposition 3 *The following are equivalent:*

1. XSSDY
2. $\mathrm{ES}_X \leq \mathrm{ES}_Y$
3. $\mathrm{FLPM}_X \leq \mathrm{FLPM}_Y$

The following are equivalent:

a. XLRY

b. $\mathrm{ES}_X \leq \mathrm{ES}_Y$ *and* $\mu_X = \mu_Y$

c. $\mathrm{FLPM}_X \leq \mathrm{FLPM}_Y$ *and* $\mu_X = \mu_Y$

d. $\Omega_X(q) \leq \Omega_Y(q)$ *iff* $q \leq \mu := \mu_X = \mu_Y$

The risk measures ES and FLPM generate an ordering of investment choices equivalent to LR, provided the means are set equal (possibly by adding or substracting the right amount of riskless prospect). In fact, ES and FLPM generate the same ordering as SSD. The ordering generated by Ω highlights a possible weakness of the Ω measure, which is that orderings between two investment choices are necessarily reversed at their common mean. This cautions against the use of the Ω ranking as an overall preference relation, unless the given sign adjustment is performed.

Denote the first crossing quantile of the two distribution functions by $\bar{q}$. More precisely, $\bar{q}$ satisfies $F_X(z) \leq F_Y(z)$ for $z < \bar{q}$, $F_X(\bar{q}) \geq F_Y(\bar{q})$ and $F_X(z) > F_Y(z)$ for $z \in (\bar{q}, \bar{q} + \epsilon)$ for some $\epsilon > 0$, and there is no smaller such crossing quantile. If there is no crossing, the results of FSD apply and we set $\bar{q} = \infty$. If there are multiple crossings but no first crossing, set $\bar{q} = -\infty$. Define $\bar{p} := F_Y(\bar{q})$.

Not all risk measures are SSD- or LR-consistent, some are SSD> or LR>consistent only, with known counterexamples to the converse:

Proposition 4 *Suppose that X* SSD*Y. Then regardless of the distribution of X and Y, the following relationships hold while while the converse is false:*

$$\begin{aligned} \mathrm{SLPM}_X &\leq \mathrm{SLPM}_Y & (7)\\ \mathrm{ZLPM}_X(q) &\leq \mathrm{ZLPM}_Y(q), \quad \forall q < \bar{q} & (8)\\ \mathrm{VaR}_X(p) &\leq \mathrm{VaR}_Y(p), \quad \forall p < \bar{p} & (9)\\ \Omega_X &\leq \Omega_Y & (10)\\ \mathrm{TCE}_X &\ ?\ \mathrm{TCE}_Y & (11) \end{aligned}$$

Suppose now that X LR*Y, then the following relationship holds while the converse is false:*

$$\sigma_X \leq \sigma_Y \tag{12}$$

Assume that X LR*Y, that $\bar{p} \in [1/4, 3/4]$ and that no other crossing point is in $[1/4, 3/4]$. Then*

$$\mathrm{IQR}_X \leq \mathrm{IQR}_Y \tag{13}$$

Starting with Porter (1974), a large literature investigates the consistency of σ, including distribution classes for which σ is not consistent. LR>consistency (and the fact that LR<consistency can fail) of SLPM is due to Porter (1974). The downside risk measures ZLPM and VaR retain partial LR>consistent ordering of assets under arbitrary asset returns distributions, partial meaning below the first crossing quantile of the two distributions. We know VaR and ZLPM are equivalent to FSD. VaR is therefore both coherent and consistent in the tail, provided there is a first crossing point. For ZLPM and VaR, we have a reversal of the ordering immediately to the right of the first crossing points. TCE is not in general LR>consistent, unless more is known about distributions, such as continuity. In conjunction with Proposition 3 we have established that Ω is LR-equivalent and SSD>consistent, but not SSD<consistent. That IQR is SSD<consistent is false even if we assume equal means.

We now introduce the overall risk measure beta. In practice, investment managers often do not choose not between the risks of two stand-alone projects, but instead between the risks of two investment choices that have themselves repercussions on the risk of an existing portfolio of assets. We have two interpretations in mind. First, assume that the market with return R is held by a well-diversified risk averse investor who then ranks the investment choices $R + X$ versus $R + Y$. We say that $X\,\mathrm{LR}_\beta Y$ if $(X + R)\mathrm{LR}(Y + R)$ and if $\sigma_X = \sigma_Y$, equivalently, if any well-diversified risk averse expected utility maximizer prefers X to Y for equal variances. The idea is that a well-diversified investor, when choosing between two investment choices with the same first two marginal moments, chooses the investment choices that lowers the overall risk of his entire portfolio. As the proposition below shows, this is indeed equivalent to the systematic risk of X –its beta β_X– being less than the systematic risk of Y. For the second interpretation, assume that investment choices X and Y have equally volatile idiosyncratic components ϵ_X and ϵ_Y respectively, where $X = \alpha_X + \beta_X R + \epsilon_X$, $E[\epsilon_X R] = E[\epsilon_X] = 0$, and similarly for Y. Then we say that $X\text{LR}'_\beta Y$ if $X\text{LR}Y$ and if $\sigma_{\epsilon_X} = \sigma_{\epsilon_Y}$, i.e. if any risk averse investor prefers, for identical idiosyncratic risk, the asset with least systematic risk. For the two-parameter environments we can show the following:

Proposition 5 *Assume that* X *and* Y *are such that* $\mu_X = \mu_Y$ *and that* $F_X(x) = F_Y(y)$ *whenever* $\frac{x-\mu_X}{\sigma_X} = \frac{y-\mu_Y}{\sigma_Y}$, *and that* $|\bar{q}| < \infty$. *The following are equivalent:*

1. $X\text{LR}Y$
2. $\sigma_X \leq \sigma_Y$
3. $\text{IQR}_X \leq \text{IQR}_Y$

Finally, are equivalent:

a. $X\text{LR}_\beta Y$

b. $\beta_X \leq \beta_Y$ *and* $\sigma_X = \sigma_Y$

as well as

i. $X\text{LR}'_\beta Y$

ii. $\beta_X \leq \beta_Y$ *and* $\sigma_{\epsilon_X} = \sigma_{\epsilon_Y}$

For two-parameter families, the risk measures σ, the IQR and β (with the additional restrictions) are SSD$-$equivalent, provided the means are equalized. The relative usefulness of the various risk measures then lies in their orderings of investment choices where neither is LR relative to each other, for instance where a chosen threshold reflects the behavior of some relevant utility function.

5 Conclusion

Risk measures are generally evaluated using a variety of desirable attributes including consistency, practicality (ease of implementation) and coherence. The use of different attributes can lead to opposing arguments regarding a risk measure. For example, VaR is relatively easy to implement but can violate consistency and coherence. This paper focuses on the consistency attribute. For the major risk measures used in practice, we extend Kaplanski and Kroll (2002) and characterize consistency with common characteristics of expected utility maximization, such as first and second order stochastic dominance.

One avenue for future work is to consider other characteristics of utility functions, such as third and higher order stochastic dominance (Eeckhoudt and Schleisinger, 2006, offer economic interpretations). Another avenue is to characterize what risk measures exhibit multiple desirable attributes, such as both consistency and coherence. Some interesting results along these lines are reported in Kusuoka (2001), De Giorgi (2005) and Leitner (2005).

A Appendix: Proofs

Proof of Proposition 1 The equivalences of 1, 2 and 3 are well-known, see Föllmer and Schied (2002).

Relationships (2) and (3) follow from Fishburn (1977). Notice that if $F_Y(z) \geq F_X(z)$ all z, then $\{z : F_X(z) \geq p\} \subseteq \{z : F_Y(z) \geq p\}$, so that the infima satisfy $\tilde{F}_X^{-1}(p) \geq \tilde{F}_Y^{-1}(p)$. Therefore $\text{ES}_Y(p) - \text{ES}_X(p) = -\frac{1}{p}\int_0^p [\tilde{F}_Y^{-1}(z) - \tilde{F}_X^{-1}(z)]dz \geq 0$, as required for (4).

As to (5), Omega is consistent since $\Omega_X \leq \Omega_Y$ iff

$$\left[\int_q^\infty [1 - F_X(z)]dz\right]\left[\int_{-\infty}^q F_Y(z)dz\right] \geq \left[\int_q^\infty [1 - F_Y(z)]dz\right]\left[\int_{-\infty}^q F_X(z)dz\right].$$

By FSD, the first term on LHS is larger than the first term on the RHS, and the same holds for the second term.

In order to prove (6), it is sufficient to provide an example whereby $X \text{FSD} Y$ and yet $\text{TCE}_X(p) > \text{TCE}_Y(p)$ for a range of p's and $\text{TCE}_X(p) < \text{TCE}_Y(p)$ over some other range. Obviously, the example must be based upon discontinuous distributions, so we choose trinomial random variables with realizations in $\{0, q, 1\}$. Consider the parameters $0 < \eta < \pi < 1$ and $0 < q < 1$. $F_Y(z) = \mathbf{1}_{0 \leq z < q}\eta + \mathbf{1}_{z \geq q}$ and $F_X(z) = \mathbf{1}_{0 \leq z < q}\eta + \mathbf{1}_{z \in [q,1)}\pi + \mathbf{1}_{z \geq 1}$, so $X \text{FSD} Y$ (strictly).

First, pick $p \in (\eta, \pi)$. We have $q_X(p) = q_Y(p) = q$, $F_X(q_X(p)) = \pi$ and $F_Y(q_Y(p)) = 1$. $\text{TCE}_Y(p) - \text{TCE}_X(p) = -q + q + q\eta - \frac{1}{\pi} q\eta = (1 - \pi^{-1}) q\eta < 0$. Now choose $p \in (\pi, 1)$. We have $q_X(p) = 1$, $q_Y(p) = q$, $F_X(q_X(p)) = 1$ and $F_Y(q_Y(p)) = 1$. $\text{TCE}_Y(p) - \text{TCE}_X(p) = -q + 1 + q\eta - (q\eta + (1-q)\pi) = (1-\pi)(1-q) > 0$. ∎

Proof of Proposition 2 That a $\Rightarrow$ b can be seen as follows. Due to $X\text{FSD}Y$, for any $z \in \mathbb{R}$, $F_X(z) \le F_Y(z) = F_X(z')$; with $z'(z) := \mu_X + \frac{\sigma_X(z-\mu_Y)}{\sigma_Y}$. Thus for any $z \in \mathbb{R}$ it holds that $z \le z'$ which is equivalent to $z\left[1 - \frac{\sigma_X}{\sigma_Y}\right] \le \mu_X - \frac{\sigma_X}{\sigma_Y}\mu_Y$. Since this must hold for all $z \in \mathbb{R}$, we have $\sigma_X = \sigma_Y$ and $\mu_Y \le \mu_X$. From here, we also see that $a \Rightarrow$ c. Now we prove b $\Rightarrow$ a. Assume $\sigma_X = \sigma_Y$ and $\mu_Y \le \mu_X$. Then $F_Y(z) = F_X(z')$ for $z' = z + (\mu_X - \mu_Y)$ since $\sigma_X = \sigma_Y$. So given any z, $F_X(z) \le F_Y(z) = F_X(z')$ iff $z \le z + (\mu_X - \mu_Y)$ iff $\mu_X - \mu_Y \ge 0$, so $X\text{FSD}Y$.

Also, $b \Leftrightarrow c$. Recall that $F_Y(z) = F_X(z')$ for z' as above. Now $q_X(3/4) - q_X(1/4) = q_Y(3/4) - q_Y(1/4)$ iff $q_X(3/4) - q_X(1/4) = (q_X(3/4) - \mu_X)\frac{\sigma_Y}{\sigma_X} + \mu_Y - (q_X(1/4) - \mu_X)\frac{\sigma_Y}{\sigma_X} + \mu_Y$ iff $q_X(3/4)\left(1 - \frac{\sigma_Y}{\sigma_X}\right) = q_X(1/4)\left(1 - \frac{\sigma_Y}{\sigma_X}\right)$. So if $q_X(3/4) \neq q_X(1/4)$, this is equivalent to $\sigma_X = \sigma_Y$. ∎

Proof of Proposition 3 That $1 \Leftrightarrow 2$ follows from a result in Föllmer and Schied (2002) (their Theorem 2.58) that says that $X\text{SSD}Y$ is equivalent to $\int_0^p \tilde{F}_X^{-1}(z)dz \ge \int_0^p \tilde{F}_Y^{-1}(z)dz$, all $p \in (0,1]$, and so also to $-\frac{1}{p}\int_0^p \tilde{F}_Y^{-1}(z)dz \ge -\frac{1}{p}\int_0^p \tilde{F}_X^{-1}(z)dz$, for all $p \in (0,1]$. $1 \Leftrightarrow 3$ is definitional.

Now turn to $a \Leftrightarrow b$. Notice that a repeated application of integration by parts shows that $E[X] = q + \int_q^\infty [1 - F_X(z)]dz - \int_{-\infty}^q F_X(z)dz$ for any $q \in \mathbb{R}$. This shows that $\Omega_X \le \Omega_Y$ iff

$$\left[\int_{-\infty}^q F_Y(z)dz\right](E[X] - q) \ge \left[\int_{-\infty}^q F_X(z)dz\right](E[Y] - q)$$

which establishes the equivalence. ∎

Proof of Proposition 4 Porter (1974) and Fishburn (1977) have established (7). Inequalities (8) and (9) follow from the fact that below the first crossing quantile $F_X(q) \le F_Y(q)$, and hence $X\text{FSD}Y$ below the first crossing quantile so that Proposition 1 can be applied below $\bar{q}$.

Inequality 10 relies on the fact that $\Omega_X \le \Omega_Y$ iff $\left(\int_{-\infty}^q F_Y(z)dz\right)(E[X]-q) \ge \left(\int_{-\infty}^q F_X(z)dz\right)(E[Y]-q)$ for all q. So if $X\text{SSD}Y$ then both items on the LHS are larger than the respective items on the RHS (recall that if XSSDY then $E[X] \geq E[Y]$: choose the utility function $u(x) = x$). The following is a counterexample to the converse. Set $0 < \eta < \pi < 1$ arbitrarily. Choose $p > 0$ arbitrarily, choose $p' > \frac{p\pi}{\pi - \eta} > p$ and set $p'' > \sup_{q \in [p,p')} \left\{ q + \frac{(q-p)\pi(1-\eta)(p'-q)}{q\eta(1-\pi)} \right\}$. Then assume X and Y are distributed as $F_X(z) = \pi \mathbf{1}_{z \in [p,p'')} + \mathbf{1}_{z \in [p'',\infty)}$ and $F_Y(z) = \eta \mathbf{1}_{z \in [0,p')} + \mathbf{1}_{z \in [p',\infty)}$. The choices of parameters above imply that $\Omega_X \leq \Omega_Y$. Now choose $q' = \lambda \frac{p\pi}{\pi-\eta} + (1-\lambda)p'$, with $\lambda \in (0,1)$. Then $q' > p$ and $q' < p'$. By construction, $\int_{-\infty}^{q'} F_X(z)dz > \int_{-\infty}^{q'} F_Y(z)dz$, so X does not SSD Y.

Relationship (11) can be shown in a similar vein to (6). We augment the example given in that proof in order to ensure $\mu_X = \mu_Y$ for a stronger result. The two random variables are now quadrinomial with distribution functions $F_Y(z) = \mathbf{1}_{0 \leq z < q}\eta + \mathbf{1}_{z \in [q,q')}\kappa + \mathbf{1}_{z \geq q'}$ and $F_X(z) = \mathbf{1}_{0 \leq z < q}\eta + \mathbf{1}_{z \in [q,q')}\pi + \mathbf{1}_{z \geq q'}$, with parameters satisfying $0 < \eta < \pi < \kappa < 1$ and realizations $0 < q < q' < 1$. First, we choose κ to ensure that $\mu_X = \mu_Y$, equivalently that $\int_0^1 [F_Y(z) - F_X(z)]dz = 0$, i.e. $\kappa = \frac{(1-q)-(1-\pi)(q'-q)}{1-q}$. It can be checked that always $\kappa \in (\pi, 1)$. Clearly, XSSDY. Now we show that for $p \in (\eta, \pi)$ we have $\text{TCE}_Y(p) - \text{TCE}_X(p) < 0$ while for $p \in (\pi, \kappa)$ we have $\text{TCE}_Y(p) - \text{TCE}_X(p) > 0$. Pick any $p \in (\eta, \pi)$. Then $q_X(p) = q_Y(p) = q$, $F_X(q_X(p)) = \pi$ and $F_Y(q_Y(p)) = \kappa$. It follows that $\text{TCE}_Y(p) - \text{TCE}_X(p) = (\kappa^{-1} - \pi^{-1})q\eta < 0$. Finally, pick $p \in (\pi, \kappa)$. Then $q_X(p) = q'$, $q_Y(p) = q$, $F_X(q_X(p)) = F_Y(q_Y(p)) = 1$. It follows that $\text{TCE}_Y(p) - \text{TCE}_X(p) = (q' - q)(1 - \pi) > 0$.

Relationship (12) is an implication of the definition of LR (use utility function $u(x) = -x^2$) and inequality (13) is immediate. ∎

Proof of Proposition 5 The results for the two parameter families are shown as follows. The proof that $\sigma_X \leq \sigma_Y$ is both necessary and sufficient for LR if there is a crossing point is due to Hanoch and Levy (1969).

The equality $\sigma_X \leq \sigma_Y \Leftrightarrow \text{IQR}_X \leq \text{IQR}_Y$ is shown as follows. Recall that $\text{IQR}_X - \text{IQR}_Y = q_X(3/4) - q_X(1/4) - q_Y(3/4) + q_Y(1/4)$. Also, by the assumption of belonging to the same two-parameter family, $q_Y(p) = \mu_Y + (q_X(p) - \mu_X)\frac{\sigma_Y}{\sigma_X}$. Substituting this into the difference equation, we get $\text{IQR}_X - \text{IQR}_Y = \left(1 - \frac{\sigma_Y}{\sigma_X}\right)\left[q_X(3/4) - q_X(1/4)\right]$.

The fact that both $X\text{LR}_\beta Y$ and $X\text{LR}'_\beta Y$ if $\beta_X \leq \beta_Y$ and respective variances are equal follows again from the above cited result in Hanoch and Levy (1969). The converse is shown as follows. For the first interpretation, notice that if $(X+R)\text{LR}(Y+R)$, then $\sigma_{X+R} \leq \sigma_{Y+R}$, which in turn is equivalent to $\beta_X \leq \beta_Y + \frac{\sigma_Y^2 - \sigma_X^2}{2\sigma_R^2} = \beta_Y$. As to the second interpretation, XLRY implies that

$\sigma_X^2 \leq \sigma_Y^2$, i.e. $\beta_X^2 \sigma_R^2 + \sigma_{\epsilon_X}^2 \leq \beta_Y^2 \sigma_R^2 + \sigma_{\epsilon_Y}^2$. With equal idiosyncratic variances, the result follows. ■

References

Acerbi, C., Nordio, C., and Sirtori, C. (2001). Expected shortfall as a tool for financial risk management. Mimeo, http://www.gloriamundi.org/var/wps.html.

Acerbi, C. and Tasche, D. (2002). On the coherence of expected shortfall. *Journal of Banking and Finance*, 26:1487–1503.

Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9 (3):203–228.

Bawa, V. (1975). Optimal rules for ordering uncertain prospects. *Journal of Financial Economics*, 2:95–121.

Brumelle, S. and Vickson, R. (1975). A unified approach to stochastic dominance. In Ziemba, W. T. and Vickson, R., editors, *Stochastic Optimization Models in Finance*. Academic Press.

De Giorgi, E. (2005). Reward-risk portfolio selection and stochastic dominance. *Journal of Banking and Finance*, 29:895–926.

Dhaene, J., Goovaerts, M. J., and Kaas, R. (2003). Economic capital allocation derived from risk measures. *North American Actuarial Journal*, 7(2):44–56.

Eeckhoudt, L. and Schleisinger, H. (2006). Putting risk in its proper place. *American Economic Review*, 96:280–289.

Fama, E. and Roll, R. (1968). Some properties of symmetric stable distributions. *Journal of the American Statistical Association*, 63:817–836.

Fishburn, P. (1977). Mean–risk analysis with risk associated with below–target returns. *American Economic Review*, 67 (2):116–126.

Föllmer, H. and Schied, A. (2002). *Stochastic Finance: An Introduction in discrete Time*. De Gruyter Studies in Mathematics.

Hanoch, G. and Levy, H. (1969). The efficiency analysis of choices involving risk. *Review of Economic Studies*, 36:335–346.

Kaplanski, G. and Kroll, Y. (2002). VaR risk measures versus traditional risk measures. *Journal of Risk*, 4(3).

Keating, C. and Shadwick, W. F. (2002). A universal performance measure. *Journal of Performance Measurement*, 6:59–84.

Kusuoka, S. (2001). On law invariant coherent risk measures. *Advances in Mathematical Economics*, 3:83–95.

Leitner, J. (2005). A short note on second-order stochastic dominance preserving coherent risk measures. *Mathematical Finance*, 15(4):649–651.

Markowitz, H. (1959). *Portfolio Selection: Efficient Diversification of Investments*. Wiley.

Porter, B. (1974). Semivariance and stochastic dominance: A comparison. *American Economic Review*, 64(1):200–204.

Quirk, J. and Saposnik, R. (1962). Admissibility and measurable utility functions. *Review of Economic Studies*, 29:140–146.

Rothschild, M. and Stiglitz, J. (1970). Increasing risk: A definition. *Journal of Economic Theory*, 2:225–243.

Tesfatsion, L. (1976). Stochastic dominance and the maximization of expected utility. *Review of Economic Studies*, 43:301–315.

Yoshiba, T. and Yamai, Y. (2002). Comparative analyses of expected shortfall and value-at-risk (2): Expected utility maximization and tail risk. *Monetary and Economic Studies*.